Differential Geometry III: Riemannian geometry

3.1 Let $\mathcal{M}^n$ be a smooth manifold and let $(x^1, \ldots, x^n)$ a local system of coordinates around $p \in \mathcal{M}$. Let also $S \in \otimes^k T_p\mathcal{M} \otimes^l T_p^*\mathcal{M}$ be a tensor of type (k, l) at p and let $S^{i_1 i_2 \ldots i_k}{}_{j_1 j_2 \ldots j_l}$ be its corresponding components. We will define the *contraction* $\mathrm{tr}(S)$ to be the tensor

$$\mathrm{tr}(S) = S^{\alpha i_2 \ldots i_k}{}_{\alpha j_2 \ldots j_l} \frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_l},$$

i.e. the components of $\mathrm{tr}(S)$ in the $(x^1, \ldots, x^n)$ coordinates are simply the components of S after summing over the first covariant and contravariant indices. Show that $\mathrm{tr}(S)$ is well-defined *independently* of the choice of coordinate system, i.e. show that if $(y^1, \ldots, y^n)$ is a different coordinate system around p and $\tilde{S}^{i_1 i_2 \ldots i_k}{}_{j_1 j_2 \ldots j_l}$ are the components of S with respect to these coordinates, then

$$\begin{aligned} S^{\alpha i_2 \ldots i_k}{}_{\alpha j_2 \ldots j_l} &\frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_l} \\ &= \tilde{S}^{\alpha i_2 \ldots i_k}{}_{\alpha j_2 \ldots j_l} \frac{\partial}{\partial y^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial y^{i_k}} \otimes dy^{j_2} \otimes \cdots \otimes dy^{j_l}. \end{aligned}$$

Remark. In the case when S is of type $(1, 1)$, and hence can be viewed as a linear map $S : T_p\mathcal{M} \to T_p\mathcal{M}$, $\mathrm{tr}(S)$ is simply the trace of the matrix representation of S; in that case, the statement of the above exercise reduces to the well-known fact that the trace of a linear automorphism is independent of the choice of basis of vectors.

Solution. Let $S^{i_1 \ldots i_k}{}_{j_1 \ldots j_l}$ and $\tilde{S}^{i_1 \ldots i_k}{}_{j_1 \ldots j_l}$ be the components of S in the $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ coordinate systems, respectively. The two sets of coordinate tangent vectors and cotangent vectors are related by

$$\frac{\partial}{\partial y^i} = \frac{\partial x^a}{\partial y^i} \quad \text{and} \quad dy^i = \frac{\partial y^i}{\partial x^a} dx^a,$$

while the relation between the two sets of components for S is given by the usual transformation law for tensors, i.e.

$$\tilde{S}^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} = S^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} \frac{\partial y^{i_1}}{\partial x^{a_1}} \cdots \frac{\partial y^{i_k}}{\partial x^{a_k}} \frac{\partial x^{b_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{b_l}}{\partial y^{j_l}}. \tag{1}$$

In the above, $\frac{\partial y^i}{\partial x^a}$ denotes the Jacobian matrix of $y = (y^1, \ldots, y^n)$ as a function of $x = (x^1, \ldots, x^1)$ (see the 1^{st} Exercise Series), while $\frac{\partial x^a}{\partial y^i}$ denotes the Jacobian of the inverse function $x = x(y)$. Recall that, for any diffeomorphism $\Phi : \Omega \subset \mathbb{R}^n \to \Omega' \subset \mathbb{R}^n$, the Jacobian matrix $\big[D(\Phi^{-1})\big]$ of the inverse function Φ^{-1} satisfies:

$$\big[D(\Phi^{-1})\big]\big(\Phi(z)\big) = \big[D(\Phi^{-1})\big]^{-1}(z) \quad \text{for all } z \in \Omega.$$

Therefore, as we've seen in class, the matrices $\left[\frac{\partial y^i}{\partial x^a}\right]$ and $\left[\frac{\partial x^a}{\partial y^i}\right]$ evaluated at the same point p in the common domain of definition of the coordinate charts $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ are the inverse of one another, i.e.

$$\frac{\partial y^i}{\partial x^a} \cdot \frac{\partial x^a}{\partial y^j} = \delta^i_j \quad \text{and} \quad \frac{\partial x^a}{\partial y^i} \cdot \frac{\partial y^i}{\partial x^b} = \delta^a_b. \tag{2}$$

In order for the contraction $\mathrm{tr}(S)$ to be well-defined independently of the coordinate system, we need to show that

$$S^{\alpha i_2\dots i_k}{}_{\alpha j_2\dots j_l}\frac{\partial}{\partial x^{i_2}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{i_2}\otimes\cdots\otimes dx^{i_l} = \tilde{S}^{\alpha i_2\dots i_k}{}_{\alpha j_2\dots j_l}\frac{\partial}{\partial y^{i_2}}\otimes\cdots\otimes\frac{\partial}{\partial y^{i_k}}\otimes dy^{i_2}\otimes\cdots\otimes dy^{i_l},$$

which is the same as saying that the components of $\mathrm{tr}(S)$ transform under changes of coordinates like a tensor of type $(k-1, l-1)$, i.e.:

$$\mathrm{tr}(\tilde{S})^{i_2\dots i_k}{}_{j_2\dots j_l} = \mathrm{tr}(\tilde{S})^{a_2\dots a_k}{}_{b_2\dots b_l}\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}}. \tag{3}$$

In order to show (3), we will calculate $\mathrm{tr}(\tilde{S})$ using the formula (1):

$$\begin{aligned}
\mathrm{tr}(\tilde{S})^{i_2\dots i_k}{}_{j_2\dots j_l} &= \tilde{S}^{\alpha i_2\dots i_k}{}_{\alpha j_2\dots j_l}\\
&= S^{a_1\dots a_k}{}_{b_1 b_2\dots b_l}\frac{\partial y^{\alpha}}{\partial x^{a_1}}\cdot\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\cdot\frac{\partial x^{b_1}}{\partial y^{\alpha}}\cdot\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}}\\
&= S^{a_1 a_2\dots a_k}{}_{b_1 b_2\dots b_l}\left(\frac{\partial y^{\alpha}}{\partial x^{a_1}}\cdot\frac{\partial x^{b_1}}{\partial y^{\alpha}}\right)\cdot\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\cdot\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}}\\
&\overset{(2)}{=} S^{a_1 a_2\dots a_k}{}_{b_1 b_2\dots b_l}\cdot\delta^{b_1}_{a_1}\cdot\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\cdot\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}}\\
&= S^{\alpha a_2\dots a_k}{}_{\alpha b_2\dots b_l}\cdot\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\cdot\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}}\\
&= \mathrm{tr}(S)^{a_2\dots a_k}{}_{b_2\dots b_l}\frac{\partial y^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\frac{\partial x^{b_2}}{\partial y^{j_2}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}},
\end{aligned}$$

i.e. (3) holds.

3.2 Let $\mathcal{M}$ be a smooth manifold of dimension n. In this exercise, we will prove that the tangent bundle $T\mathcal{M} = \cup_{p\in\mathcal{M}}T_p\mathcal{M}$ naturally admits the structure of a manifold of dimension $2n$.

Let $\{\mathcal{U}_\alpha, \phi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n\}_\alpha$ be a smooth atlas on $\mathcal{M}$. For any pair $(\mathcal{U}_\alpha, \phi_\alpha)$ in this atlas, let $(x^1, \dots, x^n)$ be the associated system of coordinates; we can define a map

$$\tilde{\phi}_\alpha : T\mathcal{U}_\alpha = \cup_{p\in\mathcal{U}_\alpha}T_p\mathcal{M} \to \phi_\alpha(\mathcal{U}_\alpha)\times\mathbb{R}^n$$

as follows:

$$\tilde{\phi}_\alpha(p, v) = (\phi(p);\, dx^1(v), \dots, dx^n(v)).$$

We will equip $T\mathcal{M}$ with the topology that makes all these maps homeomorphisms, i.e.:

$$\mathcal{V}\subset T\mathcal{M} \text{ is open iff } \tilde{\phi}_\alpha(\mathcal{V}\cap T\mathcal{U}_\alpha)\subset\mathbb{R}^n\times\mathbb{R}^n \text{ is open for all } \alpha.$$

(a) Show that $T\mathcal{M}$ equipped with the above topology is *Hausdorff* and *second countable.*

(b) Show that $\{(T\mathcal{U}_\alpha, \tilde{\phi}_\alpha)\}_\alpha$ constitutes a *smooth* atlas on $T\mathcal{M}$.

(c) Show that the base projection map $\pi : T\mathcal{M} \to \mathcal{M}$ (which acts by $\pi : T_p\mathcal{M} \to p$) is smooth. Moreover, for any smooth vector field $X \in \Gamma(T\mathcal{M})$, show that the map $X : \mathcal{M} \to T\mathcal{M}$ (sending any $p \in \mathcal{M}$ to $X_p \in T_p\mathcal{M}$) is a smooth *immersion*.

Solution. (a) In order to show that $T\mathcal{M}$ with the given topology is Hausdorff, we just have to verify that, for any two points $p_1, p_2 \in \mathcal{M}$ and tangent vectors $v_1 \in T_{p_1}\mathcal{M}$, $v_2 \in T_{p_2}\mathcal{M}$ such that $(p_1, v_1) \neq (p_2, v_2)$, there exist two disjoint open neighborhoods of (p_1, v_1) and (p_2, v_2) in $T\mathcal{M}$. We will consider two cases:

- If $p_1 \neq p_2$, then there exist open neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathcal{M}$ of p_1 and p_2, respectively, such that $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$ (since, by our definition, a manifold $\mathcal{M}$ has the Hausdorff property). Let $\phi_1 : \mathcal{U}_1' \to \mathbb{R}^n$, $\phi_2 : \mathcal{U}_2' \to \mathbb{R}^n$ be coordinate charts such that $p_1 \in \mathcal{U}_1'$ and $p_2 \in \mathcal{U}_2'$. Then the sets $\mathcal{V}_1, \mathcal{V}_2 \subset T\mathcal{M}$ defined by

$$\mathcal{V}_i = \tilde{\phi}_i^{-1}\big(\phi_i(\mathcal{U}_i \cap \mathcal{U}_i') \times \mathbb{R}^n\big) = \bigcup_{q \in \mathcal{U}_i \cap \mathcal{U}_i'} T_q\mathcal{M}, \quad i = 1, 2,$$

  have the following properties:

  1. $\mathcal{V}_i$ is open, since $\phi_i(\mathcal{U}_i \cap \mathcal{U}_i') \times \mathbb{R}^n$ is an open subset of $\mathbb{R}^{2n}$.
  2. $(p_i, v_i) \in \mathcal{V}_i$ for $i = 1, 2$, since $p_i \in \mathcal{U}_i \cap \mathcal{U}_i'$.
  3. $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ since $(\mathcal{U}_1 \cap \mathcal{U}_1') \cap (\mathcal{U}_2 \cap \mathcal{U}_2') \subseteq \mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$ and, therefore

$$\bigcup_{q \in \mathcal{U}_1 \cap \mathcal{U}_1'} T_q\mathcal{M} \cap \bigcup_{q \in \mathcal{U}_2 \cap \mathcal{U}_2'} T_q\mathcal{M} = \bigcup_{q \in (\mathcal{U}_1 \cap \mathcal{U}_1') \cap (\mathcal{U}_2 \cap \mathcal{U}_2')} T_q\mathcal{M} = \emptyset.$$

- If $p_1 = p_2 = p$, then $v_1 \neq v_2 \in T_p\mathcal{M}$. Let $\phi : \mathcal{U} \to \mathbb{R}^n$ be a coordinate chart on a neighborhood of p. Let us consifer the points $\tilde{\phi}(p, v_1)$ and $\tilde{\phi}(p, v_2)$ in $\phi(\mathcal{U}) \times \mathbb{R}^n$. Since $\tilde{\phi}$ is 1-1 and $(p, v_1) \neq (p, v_2)$, we must have $\tilde{\phi}(p, v_1) \neq \tilde{\phi}(p, v_2)$. Since $\phi(\mathcal{U}) \times \mathbb{R}^n$ is a Hausdorff space, there exist open neighborhoods $\mathcal{V}_1$, $\mathcal{V}_2$ of $\tilde{\phi}(p, v_1)$, $\tilde{\phi}(p, v_2)$, respectively, in $\phi(\mathcal{U}) \times \mathbb{R}^n$ such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$. The sets $\mathcal{U}_i = \tilde{\phi}^{-1}(\mathcal{V}_i)$, $i = 1, 2$, are then disjoint open neighborhoods of (p, v_i) in $T\mathcal{M}$.

In order to show that $T\mathcal{M}$ is second countable, we have to exhibit a countable *basis* for the topology of $T\mathcal{M}$, i.e. a countable collection $\mathcal{B}_{T\mathcal{M}} = \{\mathcal{V}_n\}_{n \in \mathbb{N}}$ of open subsets of $T\mathcal{M}$ with the property that for every non-empty open set $\mathcal{U} \subset T\mathcal{M}$ and every $z \in \mathcal{U}$, there exists a $\mathcal{V}_n \in \mathcal{B}_{T\mathcal{M}}$ such that $z \in \mathcal{V}_n$ and $\mathcal{V}_n \subset \mathcal{U}$.

Let $\mathcal{B}_{\mathcal{M}} = \{\mathcal{W}_k\}_{k \in \mathbb{N}}$ be a countable basis for the topology of $\mathcal{M}$ (such a basis always exists in view of our definition of a manifold). We can assume without loss of generality that, for each $k \in \mathbb{N}$, $\mathcal{W}_k$ is contained in the domain $\mathcal{U}_\alpha$ of at least one of the coordinate charts ϕ_α in our chosen atlas on $\mathcal{M}$ (you can easily check that, by removing from $\mathcal{B}_{\mathcal{M}}$ the sets $\mathcal{W}_k$ which are not entirely contained inside a single chart, the resulting collection is still a basis for the topology of $\mathcal{M}$). Let us make an assignment $k \to \alpha(k)$ so that $\mathcal{W}_k \subset \mathcal{U}_{\alpha(k)}$, where $\{\phi_{\alpha(k)}, \mathcal{U}_{\alpha(k)}\}$ is coordinate chart in the chosen atlas. Let also $\mathcal{B}_{\mathbb{R}^n} = \{\Omega_l\}_{l\in\mathbb{N}}$ be a countable basis for the topology of $\mathbb{R}^n$ ($\mathcal{B}_{\mathbb{R}^n}$ can be chosen to be, for instance, the set of all open balls in $\mathbb{R}^n$ of rational radius and center with rational coordinates).

The collection

$$\mathcal{B}_{T\mathcal{M}} = \big\{\tilde{\phi}^{-1}_{\alpha(k)}\big(\phi_{\alpha(k)}(\mathcal{W}_k)\times\Omega_l\big)\big\}_{(k,l)\in\mathbb{N}\times\mathbb{N}}$$

is a countable basis for the topology of $T\mathcal{M}$:

- Every set $\tilde{\phi}^{-1}_{\alpha(k)}\big(\phi_{\alpha(k)}(\mathcal{W}_k)\times\Omega_l\big)\subset T\mathcal{M}$ is open, since $\phi_{\alpha(k)}(\mathcal{W}_k)\times\Omega_l$ is an open subset of $\mathbb{R}^n\times\mathbb{R}^n$.

- Let (p,v) be a point in $T\mathcal{M}$ and $\mathcal{V}\subset T\mathcal{M}$ an open neighborhood of (p,v). Since $\mathcal{B}_{\mathcal{M}} = \{\mathcal{W}_k\}_{k\in\mathbb{N}}$ is basis of the topology of $\mathcal{M}$, there exists a $k_0\in\mathbb{N}$ such that $p\in\mathcal{W}_{k_0}$; the corresponding coordinate chart $\phi_{\alpha(k_0)}$ then covers the open neighborhood of $\mathcal{W}_{k_0}$ of p. Moreover, our definition of the topology of $T\mathcal{M}$ implies that there exists an open set $\tilde{\mathcal{V}}\subset\phi_{\alpha(k_0)}(\mathcal{W}_{k_0})\times\mathbb{R}^n$ such that $(p,v)\in\tilde{\phi}^{-1}_{\alpha(k_0)}(\tilde{\mathcal{V}})$ and $\tilde{\phi}^{-1}_{\alpha(k_0)}(\tilde{\mathcal{V}})\subset\mathcal{V}$.

  Let $\bar{\mathcal{U}}\subset\mathcal{M}$ and $\bar{\Omega}\subset\mathbb{R}^n$ be open sets containing p and $(dx^1(v),\dots,dx^n(v))$, respectively, such that

  $$\phi_{\alpha(k_0)}(\bar{\mathcal{U}})\times\bar{\Omega}\subset\tilde{\mathcal{V}}.$$

  Then, since $\mathcal{B}_{\mathcal{M}}$ and $\mathcal{B}_{\mathbb{R}^n}$ are bases of the corresponding topologies, there exist $\mathcal{W}_k\in\mathcal{B}_{\mathcal{M}}$ and $\Omega_l\in\mathcal{B}_{R^n}$ such that

  $$p\in\mathcal{W}_k,\quad ,\mathcal{W}_k\subset\bar{\mathcal{U}},\quad (dx^1(v),\dots,dx^n(v))\in\Omega_l,\quad \Omega_l\subset\bar{\Omega}.$$

  Thus, the set $\tilde{\phi}^{-1}_{\alpha(k_0)}\big(\phi_{\alpha(k_0)}(\mathcal{W}_k)\times\Omega_l\big)\subset T\mathcal{M}$ contains (p,v) and satisfies

  $$\tilde{\phi}^{-1}_{\alpha(k_0)}\big(\phi_{\alpha(k_0)}(\mathcal{W}_k)\times\Omega_l\big)\subset\tilde{\phi}^{-1}_{\alpha(k_0)}\big(\phi_{\alpha(k_0)}(\bar{\mathcal{U}})\times\bar{\Omega}\big)\subset\tilde{\phi}^{-1}_{\alpha(k_0)}(\tilde{\mathcal{V}})\subset\mathcal{V}.$$

(b) In order to show that $\big\{(T\mathcal{U}_\alpha,\tilde{\phi}_\alpha)\big\}_\alpha$ constitutes a smooth atlas on $T\mathcal{M}$, we simply have to verify that the transition maps $\Phi_{\alpha\beta}\doteq\tilde{\phi}_\alpha\circ\tilde{\phi}_\beta^{-1}:\phi_\beta(\mathcal{U}_\beta)\times\mathbb{R}^n\to\phi_\alpha(\mathcal{U}_\alpha)\times\mathbb{R}^n$ are smooth. Note that the coordinate charts $\tilde{\phi}_\alpha: T\mathcal{U}_\alpha\to\mathbb{R}^n\times\mathbb{R}^n$ can be reexpressed as follows: For any $p\in\mathcal{M}$ and $v\in T_p\mathcal{M}$,

$$\tilde{\phi}_\alpha(p,v) = (\phi_\alpha(p),\, d\phi_\alpha|_p(v))\in\mathbb{R}^n\times\mathbb{R}^n,$$

where $d\phi_\alpha|_p: T_p\mathcal{M}\to T_{\phi_\alpha(p)}\mathbb{R}^n\simeq\mathbb{R}^n$ is the differential of the map of ϕ_α. Using the formula for the derivative of the composition of two maps, we can therefore deduce that the transition map $\Phi_{\alpha\beta}$ takes the form

$$\Phi_{\alpha\beta}(z,\omega) = \big(\phi_\alpha\circ\phi_\beta^{-1}(z),\, d(\phi_\alpha\circ\phi_\beta^{-1})|_z(\omega)\big).$$

Since $\phi_\alpha\circ\phi_\beta^{-1}:\phi_\beta(\mathcal{U}_\beta)\subset\mathbb{R}^n\to\phi_\alpha(\mathcal{U}_\alpha)\subset\mathbb{R}^n$ is a C^∞ diffeomorphism (in view of the fact that $\{\mathcal{U}_\alpha,\phi_\alpha\}_\alpha$ is a smooth atlas on $\mathcal{M}$), we infer that $\Phi_{\alpha\beta}$ is a smooth map.

(c) Let $(\mathcal{U}_\alpha,\phi_\alpha)$ be a coordinate chart on $\mathcal{M}$ and let $(T\mathcal{U}_\alpha,\tilde{\phi}_\alpha)$ be the corresponding coordinate chart on $T\mathcal{M}$ as before. If $(x^1,\dots,x^n)$ are the coordinate functions associated to ϕ_α on $\mathcal{U}_\alpha$, we will denote with $(x^1,\dots,x^n;y^1,\dots,y^n)$ the corresponding coordinate functions on $T\mathcal{U}$, so that, for any $(p,v)\in T\mathcal{U}$,

$$x^i\big((p,v)\big)=x^i(p) \quad \text{and} \quad y^i\big((p,v)\big)=dx^i|_p(v).$$

In any such pair of coordinate systems on $\mathcal{M}$ and $T\mathcal{M}$, the projection map $\pi:T\mathcal{M}\to\mathcal{M}$ has the expression

$$\pi\big((x^1,\dots,x^n;y^1,\dots,y^n)\big)=(x^1,\dots,x^n).$$

Hence, π is a smooth map.

Similarly, if $X\in\Gamma(\mathcal{M})$ has components X^i with respect to the basis coordinate vector fields $\{\frac{\partial}{\partial x^j}\}_{j=1}^n$, the map $X:\mathcal{M}\to T\mathcal{M}$ defined by $p\to X_p\in T_p\mathcal{M}$ has the following expression in a pair of coordinate systems as before:

$$X\big((x^1,\dots,x^n\big)=(x^1,\dots,x^n;\,X^1(x^1,\dots,x^n),\dots,X^n(x^1,\dots,x^n)).$$

Thus, it can be readily verified that the matrix of the differential dX:

$$[dX]=\begin{bmatrix} 1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1\\ \partial_1X^1 & \partial_2X^1 & \dots & \partial_nX^1\\ \vdots & & \ddots & \\ \partial_1X^n & \partial_2X^n & \dots & \partial_nX^n \end{bmatrix}$$

has full rank, and hence X is an immersion.

3.3 Let X,Y be smooth vector fields on a smooth manifold $\mathcal{M}$. We define the commutator (or *Lie bracket*) $[X,Y]$ of X and Y to be the linear function $[X,Y]:C^\infty(\mathcal{M})\to C^\infty(\mathcal{M})$ defined by

$$[X,Y](f)=X(Y(f))-Y(X(f)) \quad \text{for all } f\in C^\infty(\mathcal{M}).$$

(a) Show that $[X,Y]$ is a smooth vector field on $\mathcal{M}$.

(b) Show that $[\cdot,\cdot]$ satisfies the following algebraic identities for any $X,Y,Z\in\Gamma(\mathcal{M})$:

1. $[X,Y]=-[Y,X]$ (*anticommutativity*).
2. $[X,aY+bZ]=a[X,Y]+b[X,Z]$ for any constans a,b ($\mathbb{R}$-*linearity*).
3. $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$ (*Jacobi identity*).

(c) Is $[\cdot,\cdot]:\Gamma(\mathcal{M})\times\Gamma(\mathcal{M})\to\Gamma(\mathcal{M})$ a $(1,2)$-tensor field?

Solution. (a) Let us first verify that, for every point $p\in\mathcal{M}$, the functional $[X,Y]_p:C^\infty(\mathcal{M})\to\mathbb{R}$, defined by

$$[X,Y]_p(f)=X_p(Y(f))-Y_p(X(f))$$

is a tangent vector at p. Since $[X,Y]_p$ is obviously a linear functional, it only remains to verify that it satisfies the product rule: For any $f,h \in C^\infty(\mathcal{M})$:

$$\begin{aligned}[X,Y]_p(f\cdot h) &= X_p(Y(f\cdot h)) - Y_p(X(f\cdot h))\\ &= X_p\big(Y(f)\cdot h + f\cdot Y(h)\big) - Y_p\big(X(f)\cdot h + f\cdot X(h)\big)\\ &= X_p(Y(f))\cdot h(p) + Y_p(f)\cdot X_p(h) + X_p(f)Y_p(h) + f(p)X_p(Y(h))\\ &\quad - Y_p(X(f))\cdot h(p) - X_p(f)\cdot Y_p(h) - Y_p(f)X_p(h) - f(p)Y_p(X(h))\\ &= \big(X_p(Y(f)) - Y_p(X(f))\big)\cdot h(p) + f(p)\cdot\big(X_p(Y(h)) - Y_p(X(h))\big),\end{aligned}$$

where, above, we made use of the fact that the functionals $X,Y : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ and $X_p, Y_p : C^\infty(\mathcal{M}) \to \mathbb{R}$ satisfy the product rule.

In order for $[X,Y]$ to be a *smooth* vector field, we have to show that the assigment $p \to [X,Y]_p$ is smooth; equivalently, we have to prove that the components of $[X,Y]$ in any local coordinate system $(x^1,\dots,x^n)$ on $\mathcal{M}$ are smooth. We can readily calculate:

$$\begin{aligned}[X,Y]^i &= [X,Y](x^i)\\ &= X(Y(x^i)) - Y(X(x^i))\\ &= X\Big(Y^j\frac{\partial x^i}{\partial x^j}\Big) - Y\Big(X^j\frac{\partial x^i}{\partial x^j}\Big)\\ &= X(Y^i) - Y(X^i)\\ &= X^j\frac{\partial Y^i}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}.\end{aligned}$$

Thus, since the components of X, Y are smooth functions, $[X,Y]$ has smooth components and is, therefore, a smooth vector field.

(b) Identities 1–3 follow easily by using the definition of $[X,Y]$.

(c) The Lie bracket $[\cdot,\cdot]$ is *not* tensorial in its arguments. Assuming, for the sake of contradiction, that it is, then it should be $C^\infty(\mathcal{M})$ -multilinear in its arguments; that is to say, for any $f \in C^\infty(\mathcal{M})$ and any $X,Y \in \Gamma(\mathcal{M})$, we should have

$$[fX,Y] = f[X,Y].$$

However, the above relation is not true: For any $h \in C^\infty(\mathcal{M})$, using the definition of $[\cdot,\cdot]$ we have:

$$\begin{aligned}[fX,Y](h) &= fX\big(Y(h)\big) - Y\big(f(X(h)\big)\\ &= f\Big(X\big(Y(h)\big) - X\big(Y(h)\big)\Big) - Y(f)X(h)\\ &= f[X,Y](h) - Y(f)X(h)\\ &\neq f[X,Y](h).\end{aligned}$$

Thus, we reach a contradiction.

3.4 Let (M,g) be a smooth Riemannian manifold.

(a) For any 1-form ω on $\mathcal{M}$, let us consider the vector field $\omega^\sharp$ defined so that, for any $X \in \Gamma(\mathcal{M})$:

$$g(X, \omega^\sharp) \doteq \omega(X).$$

Compute the components of $\omega^\sharp$ in any local coordinate chart $(x^1, \ldots, x^n)$.

(b) We will define the *gradient* of a function $f : \mathcal{M} \to \mathbb{R}$ to be the vector field

$$\nabla f \doteq df^\sharp.$$

Compute the gradient of a function $f : \mathbb{R}^2 \to \mathbb{R}$ in polar coordinates.

(c) We can naturally construct a positive definite and symmetric $(2,0)$-tensor $\tilde{g}$ acting as an inner product on the space of 1-forms by the formula

$$\tilde{g}(\omega_1, \omega_2) \doteq g(\omega_1^\sharp, \omega_2^\sharp) \quad \text{for all } \omega_1, \omega_2 \in \Gamma^*(\mathcal{M}).$$

Compute the coefficients $\tilde{g}^{ij}$ of $\tilde{g}$ in any local coordinate system as a function of the coefficients of g.

Solution. (a) In any local coordinate chart $(x^1, \ldots, x^n)$, the definition of $\omega^\sharp$ takes the form (for any $X \in \Gamma(\mathcal{M})$):

$$g_{ij} X^i (\omega^\sharp)^j = \omega_i X^i.$$

Thus, using the above formula for $X = \frac{\partial}{\partial x^k}$, $k = 1, \ldots, n$, we infer that

$$(\omega^\sharp)^i = g^{ij} \omega_j,$$

where g^{ij} are the components of the *inverse* matrix $[g]^{-1}$ of $[g]$.

(b) From part (a), we know that, in any local coordinate system $(x^1, \ldots, x^n)$ on a Riemannian manifold $(\mathcal{M}, g)$, the gradient of a function $f : \mathcal{M} \to \mathbb{R}$ takes the form

$$(\nabla f)^i = g^{ij} (df)_j = g^{ij} \frac{\partial f}{\partial x^j}.$$

In polar coordinates (r, θ) on $(\mathbb{R}^2, g_E)$, since

$$g = (dr)^2 + r^2 (d\theta)^2,$$

the components g^{ij} of $[g]^{-1}$ are

$$g^{rr} = 1, \quad g^{r\theta} = g^{\theta r} = 0, \quad g^{\theta\theta} = r^{-2}.$$

Therefore,

$$(\nabla f)^r = \frac{\partial f}{\partial r}, \quad , (\nabla f)^\theta = \frac{1}{r^2} \frac{\partial f}{\partial \theta}.$$

3.5 Let $\mathcal{M}^n$ be a smooth manifold and $\omega : \Gamma(\mathcal{M}) \to C^\infty(\mathcal{M})$ be a $C^\infty(\mathcal{M})$-linear functional. We will show that ω is in fact an 1-form on $\mathcal{M}$, i.e. if $Y \in \Gamma(\mathcal{M})$ then, for all $p \in \mathcal{M}$, $\big(\omega(Y)\big)(p)$ depends only on $Y|_p$.

(a) Explain why it suffices to show that if Y vanishes at p, then $\big(\omega(Y)\big)(p) = 0$.

(b) Let $\mathcal{U}$ be an open neighborhood of p covered by a coordinate chart $(x^1, \dots, x^n)$. Show that there exists an open neighborhood $\mathcal{V}$ of p contained inside $\mathcal{U}$ and smooth vector fields $\{X_i\}_{i=1}^n$ on $\mathcal{M}$ such that $X_i = \frac{\partial}{\partial x^i}$ on $\mathcal{V}$. (*Hint: Use a suitable cut-off function* $\psi : \mathcal{M} \to [0, +\infty)$ *which is equal to* 1 *in small a neighborhood of* p.)

(c) Show that if $Y|_p = 0$, then there exists a finite number of vector fields $\{V_k\}_k$ such that

$$Y = \sum_k f_k V_k,$$

where the functions $f_k \in C^\infty(\mathcal{M})$ satisfy $f_k(p) = 0$. Deduce that $\omega(Y)(p) = 0$.

The same argument should also work for more general $C^\infty(\mathcal{M})$-multilinear maps $T : \Gamma^*(\mathcal{M}) \times \dots \times \Gamma^*(\mathcal{M}) \times \Gamma(\mathcal{M}) \times \dots \times \Gamma(\mathcal{M}) \to C^\infty(\mathcal{M})$.

Solution. (a) In order to show that $\big(\omega(Y)\big)(p)$ depends only on $Y|_p$, we have to show that if $Y_1, Y_2 \in \Gamma(\mathcal{M})$ are such so that $Y_1|_p = Y_2|_p$, then $\big(\omega(Y_1)\big)(p) = \big(\omega(Y_2)\big)(p)$. Since ω is linear, if we set $X = Y_1 - Y_2$, the previous sentence is equivalent to the statement that if $X|_p = 0$, then $\big(\omega(X)\big)(p) = 0$.

(b) Let $\phi : \mathcal{U} \to \mathbb{R}^n$ be a local coordinate chart defined on a neighborhood $\mathcal{U}$ of p and let $(x^1, \dots, x^n)$ be the associated coordinate functions. Since $\phi(\mathcal{U})$ is an open subset of $\mathbb{R}^n$, there exists a radius $r > 0$ so that the Euclidean ball $B_{3r}(\phi(p))$ of radius $3r$ centered at $\phi(p)$ is entirely contained in $\phi(\mathcal{U})$. Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function so that

$$\chi \equiv 1 \text{ on } B_r(\phi(p)) \text{ and } \chi \equiv 0 \text{ on } \mathbb{R}^n \setminus B_{2r}(\phi(p)).$$

Let us set $\mathcal{V}_r = \phi^{-1}\big(B_r(\phi(p))\big)$, $\mathcal{V}_{2r} = \phi^{-1}\big(B_{2r}(\phi(p))\big)$ and $\mathcal{V}_{3r} = \phi^{-1}\big(B_{3r}(\phi(p))\big)$ (see Figure 1). Notice that, since ϕ is a homeomorphism, these are open subsets of $\mathcal{M}$, satisfying

$$p \in \mathcal{V}_r \subset \mathcal{V}_{2r} \subset \mathcal{V}_{3r}.$$

Moreover, since $\text{clos}\big(B_{2r}(\phi(p))\big)$ is a compact subset of $\phi(\mathcal{U})$ (since it is strictly contained inside $B_{3r}(\phi(p)) \subset \phi(\mathcal{U})$) and $\phi^{-1} : \phi(\mathcal{U}) \to \mathcal{U}$ is a homeomorphism, we know that $\text{clos}\big(B_{2r}(\phi(p))\big)$ is a compact (and, hence, closed) subset of $\mathcal{U}$. Since $\mathcal{U}$ is open, this implies in particular that

$$\partial\mathcal{U} \cap \text{clos}\big(B_{2r}(\phi(p))\big) = \emptyset. \tag{4}$$

Let us define the function $\psi : \mathcal{M} \to \mathbb{R}$ by the relation

$$\psi(q) = \begin{cases} \chi \circ \phi(q), & \text{if } q \in \mathcal{U}, \\ 0, & \text{if } q \in \mathcal{M} \setminus \mathcal{U}. \end{cases}$$

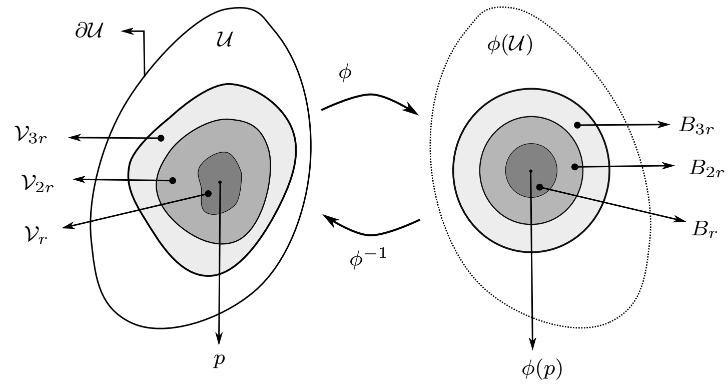

Figure 1: Schematic depiction of the subsets $\mathcal{V}_r$, $\mathcal{V}_{2r}$, $\mathcal{V}_{3r} \subset \mathcal{U}$ and $B_r(\phi(p))$, $B_{2r}(\phi(p))$, $B_{3r}(\phi(p)) \subset \mathbb{R}^n$. Note that the function ψ is supported in $\mathcal{V}_{2r}$ and $\psi \equiv 1$ on $\mathcal{V}_r$.

Note that the support of ψ is contained in the set $\mathcal{V}_{2r}$ and $\psi \equiv 1$ on $\mathcal{V}_r$. We will now show that ψ is a smooth function on $\mathcal{M}$. The definition of ψ implies that it is automatically smooth in the open sets $\mathcal{U}$ and $\mathrm{int}\big(\mathcal{M} \setminus \mathcal{U}\big)$; thus, we only have to check its behaviour at $\partial\mathcal{U}$. It will follow that $\psi \in C^\infty(\mathcal{M})$ if the set $\mathcal{Z} = \big\{q \in \mathcal{M} : \psi(q) = 0\big\}$ contains an open neighborhood of $\partial\mathcal{U}$. Indeed, since ψ is supported in V_{2r}, the set $\mathcal{Z}$ contains the open set $\mathcal{W} = \mathcal{M} \setminus \mathrm{clos}(V_{2r})$ and, in view of (4),

$$\partial\mathcal{U} \subset \mathcal{W}.$$

Having defined the smooth cut-off function $\psi : \mathcal{M} \to \mathbb{R}$ as above, let us define the vector fields X_i $(i = 1, \dots, n)$ on $\mathcal{M}$ as follows:

$$(X_i)|_q = \begin{cases} \psi(q)\frac{\partial}{\partial x^i}, & \text{if } q \in \mathcal{U}, \\ 0, & \text{if } q \in \mathcal{M} \setminus \mathcal{U}. \end{cases}$$

The vector fields X_i are indeed smooth for the same reason that ψ is smooth: They are trivially smooth on $\mathcal{U}$ and $\mathrm{int}\big(\mathcal{M} \setminus \mathcal{U}\big)$ and, since ψ vanishes on an open neighborhood of $\partial\mathcal{U}$, they are equal to the zero vector field in a neighborhood of $\partial\mathcal{U}$ (and hence they are also smooth at $\partial\mathcal{U}$). Moreover, since $\psi = 1$ on $\mathcal{V}_r$, we have

$$X_i = \frac{\partial}{\partial x^i} \quad \text{on the neighborhood } \mathcal{V}_r \text{ of } p.$$

(c) Let $Y \in \Gamma(\mathcal{M})$ be such that $Y|_p = 0$. Note that, inside the open neighborhood $\mathcal{U}$ of p covered by the coordinates $(x^1, \dots, x^n)$, we can easily write Y as a sum of vector fields with coefficients vanishing at p, since

$$Y = Y^i \frac{\partial}{\partial x^i}$$

and $Y^1(p) = \cdots = Y^n(p) = 0$ since $Y|_p = 0$. The challenge is to obtain a similar decomposition which is valid on the whole of $\mathcal{M}$ (where $\frac{\partial}{\partial x^i}$ is not well defined). To this end, we will use the cut-off function ψ and the vector fields X_i from part (b) of the exercise.

Let us first decompose (trivially)

$$Y = \psi^2 Y + (1-\psi^2)Y. \tag{5}$$

If Y^i are the components of the vector field Y in the coordinate system $(x^1, \ldots, x^n)$ on $\mathcal{U}$, then the vector field ψY can be expressed as

$$\psi(q)Y|_q = \psi(q)Y^i(q)\frac{\partial}{\partial x^i} = Y^i(q)X_i|_q \text{ for all } q \in \mathcal{U}.$$

Therefore, we have

$$\psi^2(q)Y|_q = \big(\psi Y^i\big)(q) \cdot X_i|_q \text{ for all } q \in \mathcal{U}. \tag{6}$$

Notice that, in the above expression, the vector fields $\psi^2 Y$ and X_i are defined on the whole of the manifold $\mathcal{M}$, but the functions ψY^i are only defined on $\mathcal{U}$ (covered by the coordinate system $(x^1, \ldots, x^n)$). However, for each $i = 1, ldots, n$, ψY^i vanishes in an open neighborhood of $\partial\mathcal{U}$ and hence (as in the case of ψ) it can be extended as a smooth function $h^i \in C^\infty(\mathcal{M})$ so that

$$h^i(q) = \Big\{\psi(q)Y^i(q), \quad \text{if } q \in \mathcal{U}, 0, \quad \text{if } q \in \mathcal{M}\setminus\mathcal{U}.$$

Then, since the vector field $\psi^2 Y$ satisfies (6) on $\mathcal{U}$ and vanishes identically on $\mathcal{M}\setminus\mathcal{U}$, we have

$$\psi^2 Y = h^i X_i \quad \text{everywhere on } \mathcal{M}.$$

Returning to (5), we have

$$Y = h^i X_i + (1-\psi^2)Y.$$

Notice that, on the right hand side, the coefficient of each vector field vanishes at p:

- For $i = 1, \ldots, n$, $h^i(p) = Y^i(p) = 0$ since we assumed that $Y|_p = 0$.
- $(1-\psi^2)(p) = 0$ since $\psi(p) = 1$.

Thus, we succeeded to write

$$Y = \sum_k f_k V_k$$

for $f_k \in C^\infty(\mathcal{M})$ and $V_k \in \Gamma(\mathcal{M})$ such that $f_k(p) = 0$.

In view of our assumption that $\omega(\cdot)$ is $C^\infty(\mathcal{M})$ in its argument, we therefore have:

$$\big(\omega(Y)\big)(p) = \Big(\omega\big(\sum_k f_k V_k\big)\Big)(p) = \sum_k f_k(p)\big(\omega(V_k)\big)(p) = 0.$$